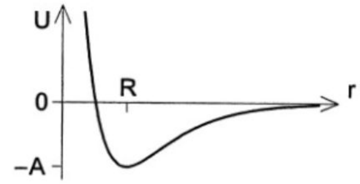
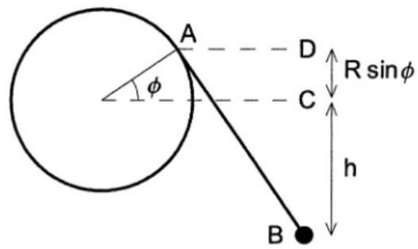
5.2 ⋆ When $r = 0$, $U = A[(e^{R/S} - 1)^2 - 1]$, which is large and positive since $R \gg S$. When $r \to \infty$, U is negative and approaches 0. The smallest possible value of U is when $r = R$ and $U = -A$; that is, the equilibrium separation is $r_o = R$. If we set $r = R + x$ and make a Taylor expansion of the exponential term in U, then



$$U = A\left[\left(\left\{1 - \frac{x}{S} + \cdots\right\} - 1\right)^2 - 1\right] \approx -A + A\left(\frac{x}{S}\right)^2 = \text{const} + \tfrac{1}{2}kx^2$$

where $k = 2A/S^2$.

5.4 ⋆⋆ The PE is $U = -mgh$ where h is the height of the mass, measured down from the level of the cylinder's center. To find h, note first that as the pendulum swings from equilibrium to angle ϕ, a length $R\phi$ of string unwinds from the cylinder. Thus the length of string away from the cylinder is $AB = (l_o + R\phi)$, and the height BD is $BD = (l_o + R\phi)\cos\phi$. Since the height $CD = R\sin\phi$, we find by subtraction that $h = BD - CD = l_o\cos\phi + R(\phi\cos\phi - \sin\phi)$. Therefore



$$U = -mgh = -mg[l_o\cos\phi + R(\phi\cos\phi - \sin\phi)].$$

If ϕ remains small we can write $\cos\phi \approx 1 - \phi^2/2$ and $\sin\phi \approx \phi$, to give

$$U \approx -mg\left\{l_o - \tfrac{1}{2}l_o\phi^2 + R\left[\phi(1 - \tfrac{1}{2}\phi^2) - \phi\right]\right\} \approx -mgl_o + \tfrac{1}{2}mgl_o\phi^2 = \text{const} + \tfrac{1}{2}k\phi^2$$

where in the third expression I dropped the term in ϕ^3. The constant $k = mgl_o$, which is the same as for a simple pendulum of length l_o. Evidently, wrapping the string around a cylinder makes no difference for small oscillations.

5.8 ⋆ **(a)** $\omega = \sqrt{k/m} = \sqrt{80/0.2} = 20\ \text{s}^{-1}$, $f = \omega/2\pi = 3.2$ Hz, and $\tau = 2\pi/\omega = 0.31$ s.

(b) Since $x_o = 0$, $A\cos(-\delta) = 0$, so $\delta = \pm\pi/2$. Since $v_o = \omega A\sin\delta = 40$ m/s, δ must be positive, $\delta = +\pi/2$, and therefore $A = v_o/\omega = 2$ m.

5.10 ⋆ If $F = -F_o\sinh\alpha x$, then $U = -\int F\,dx = (F_o/\alpha)\cosh\alpha x$. The only equilibrium position is at $x = 0$ and, for points close to this, Taylor's series gives

$$U(x) \approx (F_o/\alpha)(1 + \tfrac{1}{2}\alpha^2x^2) = \tfrac{1}{2}kx^2 + \text{const},$$

where $k = \alpha F_o$. The angular frequency of oscillations is $\omega = \sqrt{k/m} = \sqrt{\alpha F_o/m}$.

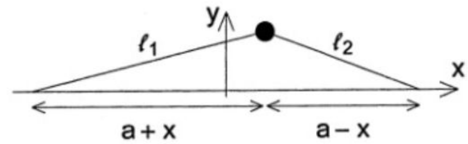
5.16 ⋆ With $\delta = \pi/2$ Eq.(5.20) reads

$$x = A_x\cos(\omega t) \qquad \text{and} \qquad y = A_y\cos(\omega t - \pi/2) = A_y\sin(\omega t)$$

from which it follows that $x^2/A_x^2 + y^2/A_y^2 = \cos^2(\omega t) + \sin^2(\omega t) = 1$, the equation of an ellipse with semi-major and semi-minor axes A_x and A_y.

I put problem 18 here because they basically have to solve this problem to do problem 19.

5.18 ⋆⋆⋆ When the mass is at position (x, y), the lengths of the two springs are l_1 and l_2, where



$$l_1 = \sqrt{(a+x)^2 + y^2} = a\left(1 + \frac{2x}{a} + \frac{x^2+y^2}{a^2}\right)^{1/2}$$

$$\approx a\left[1 + \frac{1}{2}\left(\frac{2x}{a} + \frac{x^2+y^2}{a^2}\right) - \frac{1}{8}\left(\frac{2x}{a}\right)^2\right] = a + x + \frac{y^2}{2a}.$$

Here, in passing from the first to the second line, I have used the Taylor expansion $(1+\epsilon)^{1/2} = 1 + \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + \cdots$, dropping all terms of third degree in x or y, but being careful to keep all terms of second degree. The PE of spring 1 is therefore

$$U_1 = \tfrac{1}{2}k(l_1 - l_o)^2 = \tfrac{1}{2}k[(a - l_o) + x + y^2/2a]^2$$
$$\approx \tfrac{1}{2}k[(a - l_o)^2 + 2(a - l_o)x + x^2 + (1 - l_o/a)y^2]$$

where, again, I have dropped terms of degree three in x or y. To find U_2, we have only to replace x by $-x$, and for the total PE we just add U_1 and U_2. When we do this, the terms linear in x cancel, leaving

$$U = U_1 + U_2 = k[x^2 + (1 - l_o/a)y^2] + \text{const}$$

which has the form (5.104), apart from the unimportant constant.

If $a < l_o$, the coefficient of y^2 is negative and the equilibrium at the origin O is unstable. This is because, with $a < l_o$, the springs are in compression at O. When the mass moves a little from O along the y axis, the compression in the springs forces it further away, causing the instability.

5.19 ⋆⋆⋆ The simplest way to find the total PE of all four springs is treat them two at a time. The two springs anchored on the x axis constitute the system of Problem 5.18, for which we found that

$$U_1 + U_2 = k[x^2 + (1 - l_o/a)y^2].$$

(See the solution to that problem. I've dropped an uninteresting constant here.) In exactly the same way, the PE of the two springs anchored on the y axis is

$$U_3 + U_4 = k[y^2 + (1 - l_o/a)x^2].$$

Adding these, we find for the total PE of all four springs

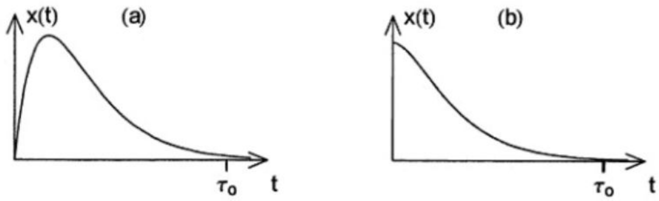
$$U = U_1 + U_2 + U_3 + U_4 = k[x^2 + y^2 + (1 - l_o/a)(x^2 + y^2)] = k\frac{2a - l_o}{a}r^2,$$

where I've used the fact that $x^2 + y^2 = r^2$. This has the advertised form $U = \frac{1}{2}k'r^2$ with an effective spring constant $k' = 2k(2a - l_o)/a$. The corresponding force is $\mathbf{F} = -\nabla U = -k'\mathbf{r}$.

5.22 ⋆ **(a)** The general solution for a critically damped oscillator ($\beta = \omega_o$) is given in (5.44) as $x(t) = e^{-\omega_o t}(C_1 + C_2 t)$. Thus

$$x_o = x(0) = C_1 \qquad \text{and} \qquad v_o = \dot{x}(0) = C_2 - \omega_o C_1 \,. \tag{ii}$$

Here $x_o = 0$, so $C_1 = 0$ and $C_2 = v_o$. Therefore, $x(t) = v_o t e^{-\omega_o t}$.



(b) In this case $v_o = 0$ and Eqs.(ii) imply that $C_1 = x_o$ and $C_2 = \omega_o x_o$. Therefore $x(t) = x_o e^{-\omega_o t}(1 + \omega_o t)$. When $t = \tau_o$, the natural period, $x = x_o e^{-2\pi}(1 + 2\pi) = 0.0136 x_o$. The motion is almost 99% damped out.

5.26 ⋆⋆ The damping changes the frequency to $\omega_1 = \sqrt{\omega_o^2 - \beta^2}$, which we can solve to give

$$\beta = \omega_o \sqrt{1 - \frac{\omega_1{}^2}{\omega_o^2}} = \omega_o \sqrt{1 - \frac{\tau_o{}^2}{\tau_1{}^2}} = \omega_o \sqrt{1 - 0.998} = 0.0447\omega_o = 0.281 \text{ s}^{-1}$$

After a time $t = 10\tau_1 \approx 10\tau_o$, the amplitude will have changed by a factor of

$$e^{-\beta t} \approx e^{-10\beta\tau_o} = e^{-20\pi\beta/\omega_o} = e^{-20\pi(0.0447)} = 0.060 \,.$$

In other words, the amplitude will have diminished by a factor of $1/0.060 = 17$. Clearly the change of amplitude of by a factor of 17 is far more noticeable than the change of period by 0.1%.

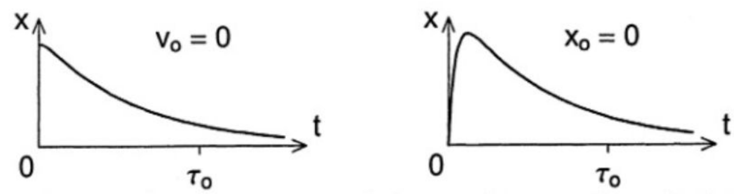
5.30 ⋆⋆ **(a)** From Eq.(5.40) we know that $x = e^{-\beta t}(C_1 e^{\lambda t} + C_2 e^{-\lambda t})$, where $\lambda = \sqrt{\beta^2 - \omega_o^2}$. We can differentiate this to get the velocity v and then set $t = 0$ to give the two equations

$$x_o = C_1 + C_2 \qquad \text{and} \qquad v_o = \lambda(C_1 - C_2) - \beta(C_1 + C_2)$$

which we can solve to give

$$C_1 = \frac{1}{2\lambda}[x_o(\lambda + \beta) + v_o] \qquad \text{and} \qquad C_2 = \frac{1}{2\lambda}[x_o(\lambda - \beta) - v_o]$$

(b)



(c) If we let $\beta \to 0$, then $\lambda \to i\omega_o$ and the coefficients C_1 and C_2 become

$$C_1 = \frac{x_o}{2} + \frac{v_o}{2i\omega_o} \qquad \text{and} \qquad C_2 = \frac{x_o}{2} - \frac{v_o}{2i\omega_o}$$

and our solution becomes

$$x = \frac{x_o}{2}\left(e^{i\omega_o t} + e^{-i\omega_o t}\right) + \frac{v_o}{2i\omega_o}\left(e^{i\omega_o t} - e^{-i\omega_o t}\right) = x_o \cos(\omega_o t) + \frac{v_o}{\omega_o}\sin(\omega_o t)$$

which you should recognize as the general solution for undamped oscillations.

5.34 ⋆ We are given that both x_p and x satisfy the same inhomogeneous equation, $Dx_p = f$ and $Dx = f$. Therefore, since D is linear, $D(x - x_p) = Dx - Dx_p = f - f = 0$. That is, the difference $x - x_p = x_h$ is a solution of the homogeneous equation $Dx_h = 0$. Therefore x can always be written as $x = x_p + x_h$ as claimed.

5.42 ⋆ The period of the pendulum is $\tau = 2\pi\sqrt{l/g} = 10.99$ s. Therefore the quality factor is $Q = \pi(\text{decay time})/\tau = \pi \times (8\text{ h})/(10.99\text{ s}) \approx 8,000$.

5.44 ⋆⋆ **(a)** Since $x = A\cos(\omega t - \delta)$, the total energy is

$$E = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}m\omega^2 A^2 \cos^2(\omega t - \delta) + \tfrac{1}{2}kA^2 \sin^2(\omega t - \delta).$$

Because $\omega \approx \omega_o$, we can replace $k = m\omega_o^2$ by $m\omega^2$, and then, since $\cos^2\theta + \sin^2\theta = 1$, we get $E = \frac{1}{2}m\omega^2 A^2$, as claimed.

(b) The rate at which the damping force dissipates energy is $F_{\text{dmp}}v = bv^2 = 2m\beta v^2$. Therefore the energy dissipated in one period is

$$\Delta E_{\text{dis}} = \int_0^\tau 2m\beta v^2 dt = 2m\beta\omega^2 A^2 \int_0^\tau \sin^2(\omega t - \delta)dt.$$

The remaining integral is just π/ω. (To see this use the trig identity $\sin^2\theta = \frac{1}{2}(1-\sin 2\theta)$ and note that the integral of the sine term over a period is zero.) Therefore, $\Delta E_{\rm dis} = 2\pi m\beta\omega A^2$.

(c) Combining the results of parts (a) and (b), we find that

$$\frac{E}{\Delta E_{\rm dis}} = \frac{\frac{1}{2}m\omega^2 A^2}{2\pi m\beta\omega A^2} = \frac{\omega_{\rm o}}{4\pi\beta} = \frac{Q}{2\pi}$$

where I have again used the fact that $\omega = \omega_{\rm o}$. That is, the ratio of the total energy to the energy lost per cycle is $Q/(2\pi)$.

5.48 ⋆⋆ If we multiply the Fourier series (5.82) by $\cos(m\omega t)$ ($m = 1, 2, 3, \cdots$) and integrate over a period, we find

$$\int \cos(m\omega t) f(t) dt = \sum_{n=0}^{\infty} a_n \int \cos(m\omega t)\cos(n\omega t) dt + \sum_{n=1}^{\infty} b_n \int \cos(m\omega t)\sin(n\omega t) dt$$

where all integrals run from $-\tau/2$ to $\tau/2$. By (5.106) every integral in the second sum is zero. By (5.105) every integral in the first sum is zero except the one with $n = m$, which is equal to $\tau/2$. Thus the whole right side collapses to a single term and

$$\int \cos(m\omega t) f(t) dt = a_m \tau/2$$

which establishes (5.83) for a_m. To establish (5.84) for b_m we do exactly the same thing except that we multiply by $\sin(m\omega t)$.

Finally, to find a_0 we just integrate the Fourier series (5.82) over a period to give

$$\int f(t) dt = \sum_{n=0}^{\infty} a_n \int \cos(n\omega t) dt + \sum_{n=0}^{\infty} b_n \int \sin(n\omega t) dt$$

where all integrals run from $-\tau/2$ to $\tau/2$. The integrals of the cosines or sines give sines or cosines, and are zero because both sines and cosines are periodic. The only exception is the integral of the cosine with $n = 0$. Since $\cos(0) = 1$, this integral is just τ, and we conclude that $\int f(t) dt = a_0 \tau$, which establishes (5.85).

5.49 ⋆⋆⋆ The given function is even, $f(-t) = +f(t)$. Therefore, $\sin(m\omega t)f(t)$ is odd and all of the integrals (5.84) for the coefficients b_m are zero. Since $\cos(m\omega t)f(t)$ is even, the coefficients a_m are not necessarily zero. Bearing in mind that $\tau = 2$, so $\omega = \pi$, we find that

$$a_0 = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} f(t)dt = \frac{f_{\max}}{2}$$

while for $m \geq 1$.

$$a_m = \frac{4}{\tau}\int_0^{\tau/2} \cos(m\omega t)f(t)dt = 2f_{\max}\int_0^1 \cos(m\pi t)(1-t)dt.$$

This integral can be evaluated (using integration by parts), and we find that

$$a_m = -\frac{2f_{\max}}{(m\pi)^2}\Big[\cos(m\pi t)\Big]_0^1 = \begin{cases} 0 & [m \text{ even}] \\ 4f_{\max}/(m\pi)^2 & [m \text{ odd}] \end{cases}$$

The left picture shows the sum of the first two terms (constant term plus first cosine) and the sawtooth function itself in gray. The right picture shows the first six terms; these follow the sawtooth so closely that it is hard to tell them apart except at the corners.